HOMOLOGY OF THE CLASSIFYING SPACE OF $Sp(n)$ GAUGE GROUPS

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ABSTRACT

We study the mod p homology of the classifying space of the gauge group associated with the principal $Sp(n)$ bundle over the four-sphere using the Serre spectral sequence for the evaluation fibration.

1. Introduction

Let G be a compact, connected simple Lie group with the classifying space BG and let P_k be the principal bundle over S^4 classified by the map $S^4 \to BG$ of degree $k \in \mathbb{Z}$. Let $\mathcal{G}_k(G)$ denote the (full) gauge group of bundle automorphisms on P_k, that is, G-equivariant self maps of P_k covering the identity map of S^4. The gauge group $\mathcal{G}_k(G)$ acts freely on the space of all G-equivariant maps from P_k to EG, $Map(P_k, EG)$, and its orbit space is given by the k-component of the space of maps from S^4 to BG, $Map_k(S^4, BG)$. Since $Map(P_k, EG)$ is contractible, the classifying space of $\mathcal{G}_k(G)$ is homotopy equivalent to $Map_k(S^4, BG)$. Similarly, if $\mathcal{G}^b_k(G)$ is the based gauge group which consists of base point preserving automorphisms on P_k, the classifying space of $\mathcal{G}^b_k(G)$ is homotopy equivalent to $\Omega^3_k G$ [2]. That is, we have

$$B\mathcal{G}_k(G) \simeq Map_k(S^4, BG), \quad B\mathcal{G}^b_k(G) \simeq \Omega^3_k G.$$

* This work was supported by Korea Research Foundation Grant KRF-2002-041-C00032.
Received January 12, 2004

Let $Sp(n)$ denote the symplectic group, that is, the group of $n \times n$ quaternionic unitary matrices. Many works consider classifying spaces of gauge groups associated with principal $Sp(1)$ bundles [1, 9, 11]. In this paper we study the mod p homology of the classifying space of the gauge group associated with the principal $Sp(n)$ bundle over S^4 using the Serre spectral sequence for the evaluation fibration.

ACKNOWLEDGEMENT: We would like to thank Fred Cohen and Mamoru Mimura for their suggestions and valuable comments.

2. Homology of $B\mathcal{G}_k^b(Sp(n))$

Let $E(x)$ be the exterior algebra on x and $\Gamma(x)$ be the divided power Hopf algebra on x which is free on generators $\gamma_i(x)$ with the product $\gamma_i(x)\gamma_j(x) = \binom{i+j}{j}\gamma_{i+j}(x)$ and with coproduct $\Delta(\gamma_n(x)) = \sum_{i=0}^{n} \gamma_{n-i}(x)\otimes\gamma_i(x)$. For a $(n+1)$-fold loop space, there are homology operations,

$$Q_{i(p-1)}\colon H_q(\Omega^{n+1}X;\mathbb{F}_p) \longrightarrow H_{pq+i(p-1)}(\Omega^{n+1}X;\mathbb{F}_p),$$

defined for $0 \le i \le n$ when $p=2$ and for $0 \le i \le n$, $i \equiv q$ for mod 2 when p is an odd prime, which is natural for a $(n+1)$-fold loop space. Let Q_i^a be the iterated operation $Q_i \cdots Q_i$ (a times) and β be the mod p Bockstein operation. We refer to [7] for the condensed treatment of these homology operations. Since $\pi_3(G) = \mathbb{Z}$ for a compact, connected simple Lie group G, $\pi_0(\Omega^3 G) = \mathbb{Z}$. Let $\Omega_0^3 G$ be the zero component of $\Omega^3 G$ and $X_{(p)}$ be the space X localized at the prime p. Note that $\Omega_0^3 G \simeq \Omega_k^3 G$ for any $k \in \mathbb{Z}$.

To get the mod p homology of $B\mathcal{G}_0^b(Sp(n))$, we compute $H_*(\Omega_0^3 Sp(n);\mathbb{F}_p)$. For the computation we need the following result, which was conjectured by Choi and Yoon [6] and proved by Lin [10].

THEOREM 2.1: *The Eilenberg-Moore spectral sequences for the path loop fibrations converging to the mod p (co)homology of the double and the triple loop spaces of any simply connected finite H-space collapse at the E_2-term.*

Hence by [8, Proposition 2.8], we have the following coalgebra isomorphism:

$$\mathrm{Tor}_{H^*(\Omega G;\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p) \cong H^*(\Omega^2 G;\mathbb{F}_p),$$
$$\mathrm{Tor}_{H^*(\Omega^2 G;\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p) \cong H^*(\Omega^3 G;\mathbb{F}_p).$$

Throughout this paper, the subscript of an element always means the degree of an element; for example, the degree of a_i is i.

THEOREM 2.2: *Let p be an odd prime. Then as an algebra, $H_*(B\mathcal{G}_0^b(Sp(n));\mathbb{F}_p)$ is isomorphic to*

$$\begin{aligned}
&\mathbb{F}_p[Q^a_{2(p-1)}(Q_{2(p-1)}[1]*[-p]) : a \geq 0]\\
&\otimes \mathbb{F}_p[Q^a_{2(p-1)}(u_{2i-2}) : a \geq 0, 1 < i \leq 2n-1, i \text{ odd}, i \not\equiv 0 \bmod p]\\
&\otimes E(Q^a_{p-1}\beta Q^{b+1}_{2(p-1)}u_{2i-2} : a,b \geq 0, \left[\frac{2n-1}{p}\right] < i \leq 2n-1, i \text{ odd}, i \not\equiv 0 \bmod p)\\
&\otimes \mathbb{F}_p[\beta Q^a_{p-1}\beta Q^b_{2(p-1)}u_{2i-2} : a,b > 0, \left[\frac{2n-1}{p}\right] < i \leq 2n-1, i \text{ odd}, i \not\equiv 0 \bmod p]\\
&\otimes E(Q^a_{p-1}Q^b_{3(p-1)}v_{2pi-3} : a,b \geq 0, \left[\frac{2n-1}{p}\right] < i \leq 2n-1, i \equiv 0 \bmod p)\\
&\otimes \mathbb{F}_p[\beta Q^{a+1}_{p-1}Q^b_{3(p-1)}v_{2pi-3} : a,b \geq 0, \left[\frac{2n-1}{p}\right] < i \leq 2n-1, i \equiv 0 \bmod p].
\end{aligned}$$

Proof: Recall the following homology:

$$H^*(Sp(n);\mathbb{F}_p) \cong E(u_{4i-1} : 1 \leq i \leq n),$$
$$\mathcal{P}^a(u_{4i-1}) = (-1)^{a(p-1)/2}\binom{2i-1}{a}u_{4i-1+2a(p-1)}.$$

Consider the Eilenberg–Moore spectral sequence converging to $H^*(\Omega Sp(n);\mathbb{F}_p)$ with

$$\begin{aligned}
E_2 &\cong \mathrm{Tor}_{H^*(Sp(n);\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p)\\
&\cong \mathrm{Tor}_{E(u_{4i-1}:1\leq i\leq n)}(\mathbb{F}_p,\mathbb{F}_p)\\
&\cong \Gamma(y_{4i-2} : 1 \leq i \leq n),
\end{aligned}$$

where $y_{4i-2} = \sigma(u_{4i-1})$. Since the E_2-term is concentrated in even degrees, the spectral sequence collapses at E_2. So $H^*(\Omega Sp(n);\mathbb{F}_p) = \Gamma(y_{4i-2} : 1 \leq i \leq n)$ as a coalgebra [8]. Now we solve algebra extension problems by the Steenrod actions. By the change of bases, we may get

$$y^p_{4i-2} = \mathcal{P}^{2i-1}(y_{4i-2}) = \binom{2i-1}{2i-1}y_{p(4i-2)} = y_{p(4i-2)}.$$

From this relation, $H^*(\Omega Sp(n);\mathbb{F}_p)$ is isomorphic as an algebra to

$$\begin{aligned}
&\mathbb{F}_p[\gamma_{p^a}(y_{2i}) : a \geq 0, 1 \leq i \leq \left[\frac{2n-1}{p}\right], i \text{ odd}, i \not\equiv 0 \bmod p]/(\gamma_{p^a}(y_{2i}))^m\\
&\otimes \Gamma(y_{2i} : \left[\frac{2n-1}{p}\right] < i \leq 2n-1, i \text{ odd}, i \not\equiv 0 \bmod p),
\end{aligned}$$

where m is a number such that $i(m-1) \leq 2n-1 < im$. Note that even though $2[\frac{2n-1}{p}] \neq [\frac{4n-2}{p}]$, we have that $\{j : [\frac{4n-2}{p}] < 2(2j-1)\} = \{j : [\frac{2n-1}{p}] < 2j-1\}$.

Consider the Eilenberg–Moore spectral sequence converging to $H_*(\Omega^2 Sp(n);\mathbb{F}_p)$ with

$$\begin{aligned}
E_2 &\cong \mathrm{Ext}_{H^*(\Omega Sp(n);\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p)\\
&\cong E(Q^a_{p-1}z_{2i-1} : a\geq 0, \left[\frac{2n-1}{p}\right] < i \leq 2n-1, i\,\mathrm{odd}, i\not\equiv 0 \bmod p)\\
&\otimes \mathbb{F}_p[\beta Q^a_{p-1}z_{2i-1} : a > 0, \left[\frac{2n-1}{p}\right] < i \leq 2n-1, i\,\mathrm{odd}, i\not\equiv 0 \bmod p]\\
&\otimes E(Q^a_{p-1}z_{2i-1} : a\geq 0, 1\leq i\leq \left[\frac{2n-1}{p}\right], i\,\mathrm{odd}, i\not\equiv 0 \bmod p)\\
&\otimes \mathbb{F}_p[Q^a_{2(p-1)}w_{2im-2} : a\geq 0, 1\leq i\leq \left[\frac{2n-1}{p}\right], i\,\mathrm{odd}, i\not\equiv 0 \bmod p].
\end{aligned}$$

By Theorem 2.1, the spectral sequence collapses at the E_2-term. After resorting, as an algebra $H_*(\Omega^2 Sp(n);\mathbb{F}_p)$ is

$$\begin{aligned}
&E(Q^a_{p-1}z_{2i-1} : a\geq 0, 1 < i \leq 2n-1, i\,\mathrm{odd}, i\not\equiv 0 \bmod p)\\
&\otimes \mathbb{F}_p[\beta Q^a_{p-1}z_{2i-1} : a > 0, \left[\frac{2n-1}{p}\right] < i \leq 2n-1, i\,\mathrm{odd}, i\not\equiv 0 \bmod p]\\
&\otimes \mathbb{F}_p[Q^a_{2(p-1)}w_{2pi-2} : a\geq 0, \left[\frac{2n-1}{p}\right] < i \leq 2n-1, i\equiv 0 \bmod p].
\end{aligned}$$

Note that under the condition $i(m-1)\leq 2n-1 < im$, we have

$$\begin{aligned}
&\{2im-2 : 1\leq i\leq [\frac{2n-1}{p}], i : \mathrm{odd}, i\not\equiv 0 \bmod p\}\\
&\qquad = \{2pi-2 : [\frac{2n-1}{p}] < i \leq 2n-1, i\equiv 0 \bmod p\}.
\end{aligned}$$

Consider the Eilenberg–Moore spectral sequence converging to $H_*(\Omega^3_0 Sp(n);\mathbb{F}_p)$ with

$$\begin{aligned}
E^2 &\cong \mathrm{Cotor}^{H_*(\Omega^2 Sp(n);\mathbb{F}_p)}(\mathbb{F}_p,\mathbb{F}_p)\\
&\cong \mathbb{F}_p[Q^a_{2(p-1)}(Q_{2(p-1)}[1]*[-p]) : a\geq 0]\\
&\otimes \mathbb{F}_p[Q^a_{2(p-1)}(u_{2i-2}) : a\geq 0, 1 < i\leq 2n-1, i\,\mathrm{odd}, i\not\equiv 0 \bmod p]\\
&\otimes E(Q^a_{p-1}\beta Q^{b+1}_{2(p-1)}u_{2i-2} : a,b\geq 0, \left[\frac{2n-1}{p}\right] < i \leq 2n-1, i\,\mathrm{odd}, i\not\equiv 0 \bmod p)\\
&\otimes \mathbb{F}_p[\beta Q^a_{p-1}\beta Q^b_{2(p-1)}u_{2i-2} : a,b > 0, \left[\frac{2n-1}{p}\right] < i \leq 2n-1, i\,\mathrm{odd}, i\not\equiv 0 \bmod p]\\
&\otimes E(Q^a_{p-1}Q^b_{3(p-1)}v_{2pi-3} : a,b\geq 0, \left[\frac{2n-1}{p}\right] < i \leq 2n-1, i\equiv 0 \bmod p)\\
&\otimes \mathbb{F}_p[\beta Q^{a+1}_{p-1}Q^b_{3(p-1)}v_{2pi-3} : a,b\geq 0, \left[\frac{2n-1}{p}\right] < i \leq 2n-1, i\equiv 0 \bmod p].
\end{aligned}$$

By Theorem 2.1, the spectral sequence also collapses at the E^2-term and we get the desired result. ∎

3. Serre spectral sequence for $H_*(B\mathcal{G}_k(Sp(n)); \mathbb{F}_p)$

Now we study the Serre spectral sequence converging to the mod p homology of the classifying space of the $Sp(n)$ gauge group for the evaluation fibration

$$\Omega^3_k Sp(n) \to Map_k(S^4, BSp(n)) \to BSp(n).$$

THEOREM 3.1: *Let $p_1^{n_1} p_2^{n_2} \cdots p_l^{n_l}$ be the prime factorization of $2n+1$. If an odd prime p is not equal to any p_i for $i = 1, \ldots, l$, then every transgression d_n: $E^n_{n,0} \to E^n_{0,n-1}$ is trivial for the Serre spectral sequence converging to $H_*(Map_k(S^4, BSp(n)); \mathbb{F}_p)$ for the evaluation fibration.*

Proof: We have the following morphisms of fibrations:

$$\begin{array}{ccccc}
\Omega^3_k Sp(n) & \longrightarrow & Map_k(S^4, BSp(n)) & \longrightarrow & BSp(n) \\
\downarrow{\scriptstyle \iota_n} & & \downarrow & & \downarrow{\scriptstyle \iota_n} \\
\Omega^3_k Sp(n+1) & \longrightarrow & Map_k(S^4, BSp(n+1)) & \longrightarrow & BSp(n+1) \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^3_k Sp & \longrightarrow & Map_k(S^4, BSp) & \longrightarrow & BSp
\end{array}$$

From the Serre spectral sequence for $\Omega^3_k Sp(n) \xrightarrow{\iota_n} \Omega^3_k Sp(n+1) \to \Omega^3 S^{4n+3}$, $(\iota_n)_*$: $H_*(\Omega^3_k Sp(n); \mathbb{F}_p) \to H_*(\Omega^3_k Sp(n+1); \mathbb{F}_p)$ is injective for all $* < 4n-1$. Similarly, $(\iota_{n+1})_*$: $H_*(\Omega^3_k Sp(n+1); \mathbb{F}_p) \to H_*(\Omega^3_k Sp(n+2); \mathbb{F}_p)$ is injective for all $* < 4n+3$, and so on. Hence ι_*: $H_*(\Omega^3_k Sp(n); \mathbb{F}_p) \to H_*(\Omega^3_k Sp; \mathbb{F}_p)$ is injective for all $* < 4n-1$, that is, every element in $H_*(\Omega^3_k Sp(n); \mathbb{F}_p)$ whose degree is less than $4n-1$ becomes stably an element in $H_*(\Omega^3_k Sp; \mathbb{F}_p)$. Note that $H_*(BSp(n); \mathbb{F}_p) \cong \Gamma(x_4, x_8, \ldots, x_{4n})$ as a coalgebra is concentrated in even degrees. Consider the Gysin sequence of the spherical fibration $S^{4n+3} \to BSp(n) \xrightarrow{i_n} BSp(n+1)$:

$$\cdots \to H_{j-4n-3}(BSp(n+1); \mathbb{F}_p) \to H_j(BSp(n); \mathbb{F}_p) \xrightarrow{(i_n)_*} H_j(BSp(n+1); \mathbb{F}_p) \to \cdots.$$

Then $(i_n)_*$ is injective for each $n \geq 1$ and $* \geq 0$. Hence i_*: $H_*(BSp(n); \mathbb{F}_p) \to H_*(BSp; \mathbb{F}_p)$ is injective for all $* \geq 0$.

Since $H_*(\Omega^3_k Sp;\mathbb{F}_p)$ and $H_*(BSp;\mathbb{F}_p)$ are concentrated in even degrees for odd primes p, the Serre spectral sequence for the bottom row fibration collapses at E_2. Since $\iota_*\colon H_*(\Omega^3_k Sp(n);\mathbb{F}_p) \to H_*(\Omega^3_k Sp;\mathbb{F}_p)$ is injective for all $* < 4n-1$, by naturality every element in $H_*(\Omega^3_k Sp(n);\mathbb{F}_p)$ whose degree is less than $4n-1$ cannot be the target of any transgression of the Serre spectral sequence for the first row fibration. That is, $d_{4i}(x_{4i}) = 0$ for $1 \le i \le n-1$. Now we claim that $d_{4n}(x_{4n}) = 0$. If $d_{4n}(x_{4n})$ is not trivial, then the transgression from it is the first non-trivial differential. Since the target of the first non-trivial differential is primitive in such a spectral sequence, $d_{4n}(x_{4n})$ is a primitive element of degree $4n-1$ in $H_*(\Omega^3_k Sp(n);\mathbb{F}_p)$.

Now we check odd degree primitive elements in $H_*(\Omega^3_k Sp(n);\mathbb{F}_p)$ from Theorem 2.2. The element v_{2pi-3} is defined for odd i satisfying the condition that $[(2n-1)/p] < i \le 2n-1, i \equiv 0 \bmod p$. Since $2n+1 = p_1^{n_1}p_2^{n_2}\cdots p_l^{n_l}$ and p is not equal to any p_i for $i = 1,\ldots,l$, the number $p_1^{n_1}p_2^{n_2}\cdots p_l^{n_l}/p$ cannot be an integer. Moreover, $[(p_1^{n_1}p_2^{n_2}\cdots p_l^{n_l}-2)/p] = [p_1^{n_1}p_2^{n_2}\cdots p_l^{n_l}/p]$, that is, $[(2n-1)/p] = [(2n+1)/p]$. So

$$|v_{2pi-3}| > 2p(2n+1)/p - 3 = 4n-1.$$

Now we consider the element u_{2i-2} which is defined for odd i with $[(2n-1)/p] < i \le 2n-1, i \not\equiv 0 \bmod p$. We divide into two cases.

(Case 1) $p < 2n+1$. Since $[(2n-1)/p] \ge 1$ and $[(2n-1)/p] = [(2n+1)/p]$,

$$|\beta Q_{2(p-1)}u_{2i-2}| > p(2(2n+1)/p - 2) + 2(p-1) - 1 = 4n-1.$$

(Case 2) $p > 2n+1$. Then $(2n-1)/p < 1$. Since $[(2n-1)/p] < i \le 2n-1, i\,\text{odd}, i \not\equiv 0 \bmod p$, i can be 1. $|\beta Q_{2(p-1)}u_0| = 2p-3$. Since $p > 2n+1$,

$$|\beta Q_{2(p-1)}u_0| > 2(2n+1) - 3 = 4n-1.$$

Hence there is no primitive element of dimension $4n-1$ in $H_*(\Omega^3_k Sp(n);\mathbb{F}_p)$ for p which is not equal to any p_i for $i = 1,\ldots,l$. Therefore $\tau(x_{4n}) = 0$. Consider the following morphism of fibration sequences up to homotopy:

$$\begin{array}{ccccc} Sp(n) & \longrightarrow & * & \longrightarrow & BSp(n) \\ \downarrow{\scriptstyle \partial_k} & & \downarrow & & \downarrow \\ \Omega^3_k Sp(n) & \longrightarrow & Map_k(S^4, BSp(n)) & \longrightarrow & BSp(n) \end{array}$$

From the above argument, $(\partial_k)_*\colon H_*(Sp(n);\mathbb{F}_p) \to H_*(\Omega^3_k Sp(n);\mathbb{F}_p)$ is a zero map, so that every transgression $d_{4t}\colon H_{4t}(BSp(n);\mathbb{F}_p) \to H_{4t-1}(\Omega^3_k Sp(n);\mathbb{F}_p)$ for $t \ge 1$ is trivial. ∎

Remarks: 1. For $p = 2$, every transgression is also trivial for the Serre spectral sequence converging to $H_*(Map_k(S^4, BSp(n)); \mathbb{F}_2)$ for the evaluation fibration [5].

2. For $Sp(1)$, consider the following sequence of fibrations up to homotopy:

$$\cdots \to Sp(1) \to \Omega^3 Sp(1) \to Map(S^4, BSp(1)) \to BSp(1) \to \cdots.$$

We have $[Sp(1), \Omega^3 Sp(1)] = \pi_6(S^3) = \mathbb{Z}/(3) \oplus \mathbb{Z}/(4)$. Hence localized at $p > 3$, any map $f\colon Sp(1) \to \Omega^3 Sp(1)$ is null homotopic. From this, the Serre spectral sequence converging to $H_*(Map_k(S^4, BSp(1)); \mathbb{F}_p)$ for the evaluation fibration collapses at the E^2-term for all $p > 3$. Hence if $p \neq 3$, there is an isomorphism of $\mathbb{F}_p$-vector spaces for all integers k

$$H_*(B\mathcal{G}_k(Sp(1)); \mathbb{F}_p) \cong H_*(\Omega^3_k Sp(1); \mathbb{F}_p) \otimes H_*(BSp(1); \mathbb{F}_p).$$

3. From the proof of Theorem 3.1, we obtain that

$$q_*\colon H_*(Map_k(S^4, BSp(n)); \mathbb{F}_p) \to H_*(BSp(n); \mathbb{F}_p)$$

is onto. But we do not know whether the fiber $\Omega^3_k Sp(n)$ is totally nonhomologous to zero in the total space with respect to a field $\mathbb{F}_p$.

4. Non-collapsing case

Now we turn to the case for the prime $p = 2n + 1$. Consider the exact sequence of homotopy groups:

$$\cdots \to \pi_{4n-1}(Sp(n)) \xrightarrow{(\partial_k)_\#} \pi_{4n-1}(\Omega^3_k Sp(n)) \to \pi_{4n-1}(Map_k(S^4, BSp(n)) \to \cdots.$$

Then the boundary map $(\partial_k)_\#$ can be expressed in terms of the Samelson product $<,>$ as follows [13, 15]. For α in $\pi_{4n-1}(Sp(n)$, we have

$$(\partial_k)_\# \alpha = \pm k \langle \alpha, \beta \rangle \tag{1}$$

where β generates $\pi_3(Sp(n))$ [13, Proposition 2.1]. We recall the following fact.

PROPOSITION 4.1 ([4, Theorem 2]): *The kernel of the homomorphism*

$$\pi_{4n-1}(Sp(n)) \otimes \pi_{4m-1}(Sp(m)) \to \pi_{4n+4m-2}(Sp(n+m-1))$$
$$\alpha \otimes \beta \mapsto \langle \alpha, \beta \rangle$$

induced by the Samelson product $\langle , \rangle$ is precisely divisible by $k_{n+m}/k_n k_m$ where

$$k_r = \begin{cases} (2r-1)!2, & r \text{ even}, \\ (2r-1)!, & r \text{ odd}. \end{cases}$$

Since $\pi_k(Sp) = \pi_k(Sp(n))$ for $n \geq (k-1)/4$ from the stability [3], we have

$$\pi_{4n-1}(Sp(n)) = \pi_{4n-1}(Sp(m+n-1))$$

and

$$\pi_{4m-1}(Sp(m)) = \pi_{4m-1}(Sp(m+n-1)).$$

Hence for $m = 1$ we have

$$\begin{array}{ccc} \pi_{4n-1}(Sp(n)) \otimes \pi_3(Sp(n)) & \to & \pi_{4n+2}(Sp(n)) \\ \alpha \otimes \beta & \mapsto & \langle \alpha, \beta \rangle. \end{array}$$

By Proposition 4.1, the order of the element $\langle \alpha, \beta \rangle$ is $n(2n+1)$ for even n and $(2n+1)4n$ for odd n for generators α, β.

Hence if $p = 2n+1$, then $\partial_k \colon Sp(n)_{(p)} \to \Omega_k^3 Sp(n)_{(p)}$ is not null homotopic for $k \not\equiv 0 \bmod p$ by (1).

Recall the p-primary component of homotopy groups of odd spheres [14, p. 176].

PROPOSITION 4.2: *Let p be an odd prime. Then we have the following.*

$$\pi_{2m+1+2i(p-1)-2}(S^{2m+1};p) = \mathbb{Z}/(p) \quad \text{for } 1 \leq m < i, \text{ and } i = 2, \ldots, p-1.$$
$$\pi_{2m+1+2i(p-1)-1}(S^{2m+1};p) = \mathbb{Z}/(p) \quad \text{for } 1 \leq m, \text{ and } i = 1, 2, \ldots, p-1.$$
$$\pi_{2m+1+k}(S^{2m+1};p) = 0 \text{ otherwise} \quad \text{for } k < 2p(p-1)-2.$$

Now we prove the following theorem.

THEOREM 4.3: *Let $2n+1$ be an odd prime. If $p = 2n+1$, then we have the following transgression:*

$$d_{4n} \colon E_{4n,0}^{4n} \to E_{0,4n-1}^{4n} = \begin{cases} 0, & k \equiv 0 \bmod p \\ \text{nonzero}, & k \not\equiv 0 \bmod p \end{cases}$$

for the Serre spectral sequence converging to $H_*(Map_k(S^4, BSp(n)); \mathbb{F}_p)$ of the evaluation fibration. Hence the Serre spectral sequence does not collapse at the E^2-term for $k \not\equiv 0 \bmod p$.

Proof: $Sp(n)$ is p-regular if and only if $2n \leq p$ [12, p. 293]. So if $p = 2n+1$, $Sp(n)$ is p-regular. So there is a p-equivalence

$$Sp(n) \simeq_p S^3 \times S^7 \times \cdots \times S^{4n-1}.$$

Hence localized at p, $\partial_k \colon S^3 \times S^7 \times \cdots \times S^{4n-1} \to \Omega_k^3 S^3 \times \Omega^3 S^7 \times \cdots \times \Omega^3 S^{4n-1}$ is not null homotopic for $k \not\equiv 0 \bmod p$ by the above argument.

Consider $\pi_{4i-1}(\Omega^3 S^{4j-1};p) = \pi_{4i+2}(S^{4j-1};p)$ for $1 \le i,j \le n$. Then by Proposition 4.2, we have

$$\pi_{4i+2}(S^{4j-1};p) = \begin{cases} \mathbb{Z}/(p), & \text{if } i=n \text{ and } j=1, \\ 0, & \text{otherwise for } 1 \le i,j \le n. \end{cases}$$

Hence the map

$$p_1 \circ \partial_k|_{S^{4n-1}}\colon S^{4n-1} \xrightarrow{\partial_k|_{S^{4n-1}}} \Omega^3_k S^3 \times \Omega^3 S^7 \times \cdots \otimes \Omega^3 S^{4n-1} \xrightarrow{p_1} \Omega^3_k S^3$$

is not null. Let $k \not\equiv 0 \bmod p$ and $p_1 \circ \partial_k|_{S^{4n-1}} = g$. Then $g_\#\colon \pi_{4n-1}(S^{4n-1} : p) \to \pi_{4n-1}(\Omega^3_k S^3 : p)$ is not zero. Since $\pi_1(\Omega^3_k S^3) = \mathbb{Z}/2\mathbb{Z}$, we have the following fibration:

$$(\Omega^3_k S^3)\langle 1\rangle \xrightarrow{j} \Omega^3_k S^3 \to K(\mathbb{Z}/2\mathbb{Z},1).$$

Since $\pi_{4n-1}(K(\mathbb{Z}/2\mathbb{Z},1);p) = 0$, there is a map $h\colon S^{4n-1} \to (\Omega^3_k S^3)\langle 1\rangle$ such that $j \circ h = g$. Consider $h\colon S^{4n-1} \to (\Omega^3_k S^3)\langle 1\rangle$. Then $h_\#\colon \pi_i(S^{4n-1};p) \to \pi_i((\Omega^3_k S^3)\langle 1\rangle;p)$ is an isomorphism for $i \le 4n-1$. Note that $\pi_{3+i}(S^3;p) = 0$ for $0 < i < 4n-1$ by Proposition 4.2. By the mod p J. H. C. Whitehead Theorem, we have $h_*\colon H_i(S^{4n-1};\mathbb{F}_p) \to H_i((\Omega^3_k S^3)\langle 1\rangle;\mathbb{F}_p)$ is an isomorphism for $i \le 4n-1$. That is, $h_*\colon H_{4n-1}(S^{4n-1};\mathbb{F}_p) \to H_{4n-1}((\Omega^3_k S^3)\langle 1\rangle;\mathbb{F}_p)$ is an isomorphism. So $g_*\colon H_{4n-1}(S^{4n-1};\mathbb{F}_p) \to H_{4n-1}(\Omega^3_k S^3;\mathbb{F}_p)$ is also an isomorphism for $k \not\equiv 0 \bmod p$. Hence we get that

$$(\partial_k)_* = \begin{cases} 0, & k \equiv 0 \bmod p, \\ \text{nonzero}, & k \not\equiv 0 \bmod p. \end{cases} \tag{2}$$

Now we consider the Serre spectral sequence for the following fibration:

$$\Omega^3_k Sp(n) \to Map_k(S^4, BSp(n)) \to BSp(n).$$

From the same argument of Theorem 3.1, we have transgressions $d^k_{4i} = 0$ for $1 \le i \le n-1$ and the first possible non-trivial differential is

$$d^k_{4n}\colon H_{4n}(BSp(n);\mathbb{F}_p) \to H_{4n-1}(\Omega^3_k Sp(n));\mathbb{F}_p).$$

By naturality of differentials, we have $d^k_{4n} = (\partial_k)_* \circ d_{4n}$ for the following morphism of fibration sequences up to homotopy:

$$\begin{array}{ccccc} Sp(n) & \longrightarrow & * & \longrightarrow & BSp(n) \\ {\scriptstyle \partial_k}\downarrow & & \downarrow & & \| \\ \Omega^3_k Sp(n) & \longrightarrow & Map_k(S^4, BSp(n)) & \longrightarrow & BSp(n) \end{array}$$

From (2), we get

$$d^k_{4n} = \begin{cases} 0, & k \equiv 0 \bmod p, \\ \text{nonzero}, & k \not\equiv 0 \bmod p. \end{cases}$$

Hence the Serre spectral sequence for the evaluation fibration does not collapse at the E^2-term for $k \not\equiv 0 \bmod p$. ∎

Remark: Let p be an odd prime greater than $2n+1$. Then by Proposition 4.2, we have

$$\pi_{4i+2}(S^{4j-1};p) = 0 \quad \text{for all } 1 \le i,j \le n.$$

Hence $(\partial_k)_\#$ is trivial for any integer k. So we obtain the following isomorphism for all $i \ge 1$:

$$\begin{aligned}\pi_i(Map(S^4, BSp(n));p) &\cong \pi_i(\Omega^3 Sp(n);p) \oplus \pi_i(BSp(n);p) \\ &\cong \pi_{i+3}(Sp(n);p) \oplus \pi_{i-1}(Sp(n);p) \\ &\cong \bigoplus_{j=1}^{n} (\pi_{i+3}(S^{4j-1};p) \oplus \pi_{i-1}(S^{4j-1};p)).\end{aligned}$$

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